Illustrating amazing effects of optics with the computer

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***Abstract*—Optical systems may be complex to study, especially when they involve media with spatially varying refractive index. A fast, accurate and easy to use MATLAB code for solving the iconal equation in such media is presented. It is used for ray-tracing the propagation of light in non-homogeneous media and illustrating some amazing effects in modern physics that cannot be brought to the attention of students without the aid of numerical simulations.**

***Index Terms*—optics, ray-tracing, non-homogeneous media.**

I. INTRODUCTION

Ray optics is the branch of optics in which all the subtle wave effects are neglected: the light is considered as travelling along rays which can only change their direction by refraction or reflection. On one hand, when light propagates in media with constant refractive index, the SNELL-DESCARTES laws can be applied for implementing a fast numerical ray-tracing procedure based on a geometrical approach of the problem. On the other hand, when propagation takes place in media with non-homogeneous refractive index, the differential equation governing the propagation of light has to be solved with the aid of the computer. This contribution describes a fast, accurate and easy to use code for illustrating the propagation of light in such media, sometimes with amazing effects that cannot be brought to the attention of students without the aid of numerical simulations, or except at an expensive cost. It has been developed under MATLAB environment in the framework of an educational project, but it is general enough to be useful in most of the cases. All the lines of code are given so that they can be freely distributed and re-used.

II. SOLVING THE ICONAL EQUATION

With the incorporation of computers in the curriculum, it has become much easier to bring numerical simulations into the classroom. Since MATLAB is widely used in french universities, it is a natural choice for the authors. However, before delving into the details of the code, it is necessary to understand why the study of the propagation of light in some media sometimes raises difficulties that cannot be solved without the aid of the computer.

The iconal equation, which describes the path of the light propagating in a medium with refractive index n, is an ordinary differential equation (ODE) of the second order [1]:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}\right) = \nabla n, \tag{1}$$

where $\mathbf{r}$ denotes the position vector of a point on the ray and $\mathrm{d}s$ is an element of length along the path. The classical approach for solving (1) is to set $f = \mathrm{d}\mathbf{r}/\mathrm{d}s$; however from the numerical implementation of a ray tracing procedure point of view it is more convenient to set $\mathrm{d}\ell = \mathrm{d}s/n$ so that (1) now reads [2]:

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}\ell^2} = \frac{1}{2}\nabla n^2. \tag{2}$$

When light is propagating in a plane, the vector equation (2) reduces to a system of two scalar equations of the second order. We therefore have in cartesian coordinates:

$$\left\{\begin{array}{ccc} \dfrac{\mathrm{d}^2x}{\mathrm{d}\ell^2} & = & \dfrac{1}{2}\dfrac{\partial n^2}{\partial x}, \\ \dfrac{\mathrm{d}^2y}{\mathrm{d}\ell^2} & = & \dfrac{1}{2}\dfrac{\partial n^2}{\partial y}. \end{array}\right. \tag{3}$$

As soon as the refractive index does not vary with x and y, (3) is straiforward to integrate and leads to a straight line path. When n is not uniform and according to the dependency of n with respect to x and y, it may not be possible to integrate (3) without the aid of the computer. This is why the capabilities of MATLAB [3] to quickly solve ODE is used here for studying the propagation of light in non-homogeneous media. However, since MATLAB can only solve ODE of the first order, (3) has to be rewritten:

$$\left\{\begin{array}{ccc} \dfrac{\mathrm{d}p_x}{\mathrm{d}\ell} & = & \dfrac{1}{2}\dfrac{\partial n^2}{\partial x}, \\ \dfrac{\mathrm{d}p_y}{\mathrm{d}\ell} & = & \dfrac{1}{2}\dfrac{\partial n^2}{\partial y}, \end{array}\right. \quad \text{with} \quad \begin{array}{ccc} \dfrac{\mathrm{d}x}{\mathrm{d}\ell} & = & p_x, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\ell} & = & p_y. \end{array} \tag{4}$$

The MATLAB function `iconalODE` contains the final definition of the problem under study: in vector `f` are stored the values of x, y, p_x and p_y for a given value of ℓ, whereas the vector `df` returns the values of p_x, p_y, $\mathrm{d}p_x/\mathrm{d}\ell$ and $\mathrm{d}p_y/\mathrm{d}\ell$ for the same value of ℓ.

```
function df=iconalODE(l,f)
x = f(1);  px = f(3);
y = f(2);  py = f(4);
[dn2dx, dn2dy] = dn2(x,y,param);
dpxdl = 0.5*dn2dx;
dpydl = 0.5*dn2dy;
df = [px; py; dpxdl; dpydl];
```

The student has to supply a function `dn2` which should return the cartesian components of the gradient of n^2 at any point (x, y), the expression of which may depend on parameters stored in the `param` vector. The equation described in `iconalODE` is solved with the aid of the built-in function `ode45` whose integration scheme is based on a fifth order RUNGE-KUTTA approach [4]:

```
>> [l,f] = ode45('iconalODE',l,fi);
>> X = f(:,1);
>> Y = f(:,2);
>> dYdX = f(:,4)./f(:,3);
```

The solver is fed with a vector `fi` which contains the initial settings of the problem, namely x_i, y_i, $(\mathrm{d}x/\mathrm{d}\ell)_i$ and $(\mathrm{d}y/\mathrm{d}\ell)_i$ at a given starting point M_i. The values of the solution, namely x, y, $\mathrm{d}x/\mathrm{d}\ell$ et $\mathrm{d}y/\mathrm{d}\ell$ along the ray path, are returned in the array `f`: here, for convenience reasons, vectors `X`, `Y` and `dYdX` are used for storing the values of x, y and dy/dx.

III. APPLICATION IN NON-HOMOGENEOUS MEDIA

In this study, the refractive index of the non-homogeneous media satisfies the law:

$$n^2(\rho) = 1 + \frac{\rho_o^2}{\rho^2} \quad \text{with} \quad \rho = \sqrt{x^2+y^2}, \tag{5}$$

which is inspired from previous work on relativistic particles in a KEPLER or COULOMB potential [5]. Another choice would have been to study the propagation of light in LUNEBURG lens [6], however contrary to expression (5) the refractive index would have not present a singularity and consequently the example would have been less constraining from the computational point of view and less illustrative from the point of view of the propagation of light in non-homogeneous media. However, according to the approach adopted by the authors, moving from one study to another just requires to change the lines of code written in the function `dn2` supplied by the student: this is exactly what is done in a classroom.

Coming back to (5), the region where the media is non-homogeneous will be restricted to a disk $\mathcal{D}$ with radius R. Outside $\mathcal{D}$, n will be supposed to be uniform and equal to:

$$n_i = \sqrt{1+\rho_o^2/R^2}, \tag{6}$$

so that no discontinuity occurs at the boundary. Within $\mathcal{D}$, the components of the gradient of n^2 in cartesian coordinates are:

$$\begin{aligned} \frac{\partial n^2}{\partial x} &= -\frac{2x\rho_0^2}{(x^2+y^2)^2}, \\ \frac{\partial n^2}{\partial y} &= -\frac{2y\rho_0^2}{(x^2+y^2)^2}, \end{aligned} \tag{7}$$

whereas they are null outside $\mathcal{D}$ since n is there uniform and equal to n_i. The function `dn2` is therefore reduced to:

```
function [dn2dx,dn2dy]=dn2(x,y,param)
Ro = param(1);
R  = param(2);
if (sqrt(x^2+y^2) > R)
    dn2dx = 0;
    dn2dy = 0;
else
    dn2dx = -2*x*Ro^2/(x^2+y^2)^2;
    dn2dy = -2*y*Ro^2/(x^2+y^2)^2;
end
```

where `Ro` is used for storing the value of parameter ρ_o and `R` that of the radius R of $\mathcal{D}$.

Before solving the ODE described in functions `iconalODE` and `dn2` it is necessary to set the initial conditions of the problem. Let us consider an incident ray starting at a point $M_i(x_i, y_i)$ and making an angle α with Ox axis. In the neighbourhood of M_i, we have $\mathrm{d}x = \mathrm{d}s\cos\alpha$ and $\mathrm{d}y = \mathrm{d}s\sin\alpha$. Since at any point on the ray path $\mathrm{d}\ell = \mathrm{d}s/n$, we therefore have:

$$(\mathrm{d}x/\mathrm{d}\ell)_i = n_i\cos\alpha \quad \text{and} \quad (\mathrm{d}y/\mathrm{d}\ell)_i = n_i\sin\alpha. \tag{8}$$

Finally, the radius of $\mathcal{D}$ is set, for example, to $R = 4\rho_o$ with $\rho_o = 1$. Since the system exhibits a central symmetry, the study is restricted here to incidents rays parallel to Ox axis, so that $\alpha = \pi$, and starting from M_i located at a distance $y_i = h = 2\rho_o$ from Ox axis and with $x_i = 6\rho_o$. The following lines are the MATLAB code which corresponds to all these initial settings.

```
>> Ro = 1;
>> R = 4*Ro;
>> h = 2*Ro;
>> Xi = 6*Ro;
>> Yi = h;
>> alpha = pi;
>> Ni = sqrt(1 + (Ro/R)^2);
>> dXidl = Ni*cos(alpha);
>> dYidl = Ni*sin(alpha);
>> fi = [Xi Yi dXidl dYidl];
```

After point M_i, the path of the light is the solution of the ODE described in functions `iconalODE` and `dn2`. The complete path of the ray returned by the `ode45` solver, as described in the previous section, is represented on Fig. 1. It corresponds to the following lines:

```
>> figure(1);
>> plot(X/Ro,Y/Ro,'r-');
```

As expected, outside the disk $\mathcal{D}$, the light is travelling in

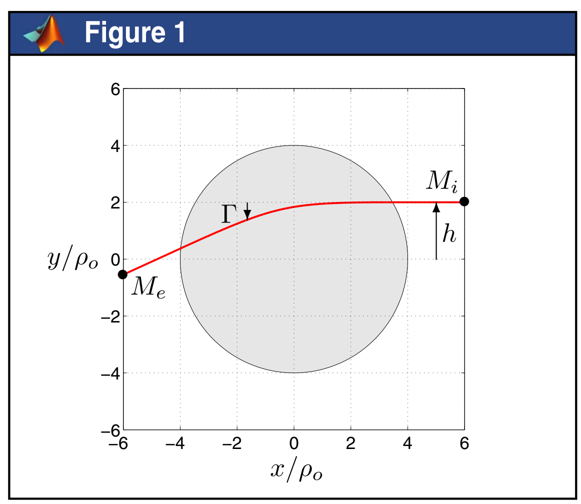

Fig. 1. An example of a ray path (in red) between points M_i and M_e. Outside the disk $\mathcal{D}$ (in grey) the media is homogeneous with a constant refractive index: the light is travelling in straight line. On the contrary, the media in $\mathcal{D}$ is non-homogeneous with spatially varying refractive index: the path of the light is curved.

straight line since the media is homogeneous with uniform refractive index (6). On the contrary, within $\mathcal{D}$ the path is curved since the light is here travelling in a non-homogeneous media with spatially varying refractive index (5). The curvature of the path is oriented towards the center of the disk $\mathcal{D}$, that is to say towards the direction of the gradient of n. The ray continuously tends towards the center without reaching it and moves away in a symetric manner with respect to the location where the distance from the center of $\mathcal{D}$ was minimal. The deviation angle Γ after travelling through the non-homogeneous media can be computed at any point M_e in the homogeneous media according to:

$$\Gamma = \arctan(\mathrm{d}y/\mathrm{d}x)_e + \pi$$

that is to say in MATLAB language:

```
>> gamma = atan(dYdX(end))+pi
```

As observed by students, h is playing the role of an impact parameter with respect to the singularity of the refractive index at the center of the disk $\mathcal{D}$ and the angle Γ that of a scattering angle with regards to the direction of the incoming ray. Shown in Fig. 1, for $h = 2\rho_o$, the deviation is about 25° with respect to the incident direction. An interesting and convenient aspect of conducting numerical simulations with the aid of the computer in a classroom with MATLAB is the capability to re-run the code with different initial settings and to obtain the new path of the light almost immediately. This is exactly what is done by the student for investigating the influence of the parameter h on the deviation angle Γ. As shown in Fig. 2, Γ varies with h in a non-linear manner. Values of Γ greater than $\pi/2$ correspond to rays which make a half turn, or even more than a complete turn, before leaving the non-homogeneous media.

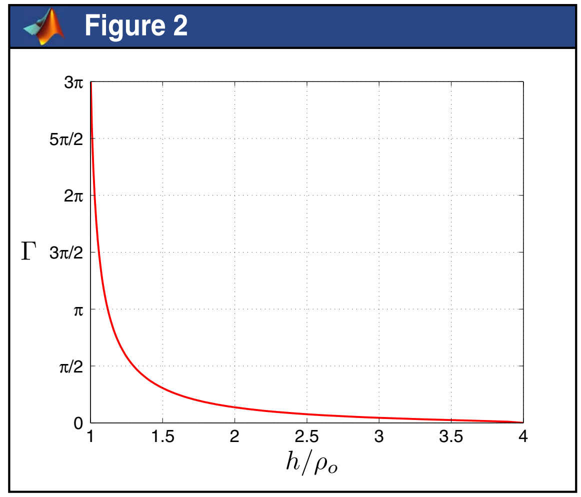

Fig. 2. Variations of the deviation angle Γ with the parameter h.

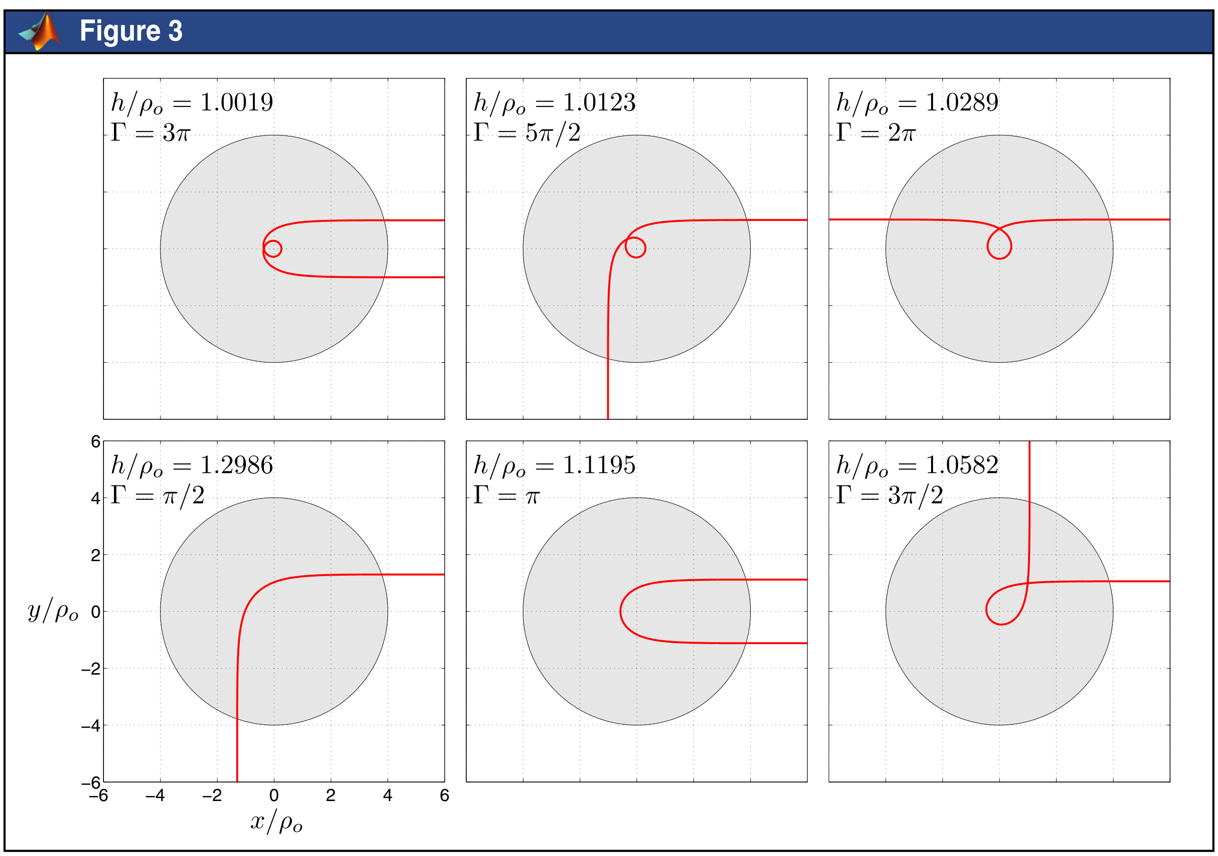

Fig. 3. Some light trajectories for various values of h.

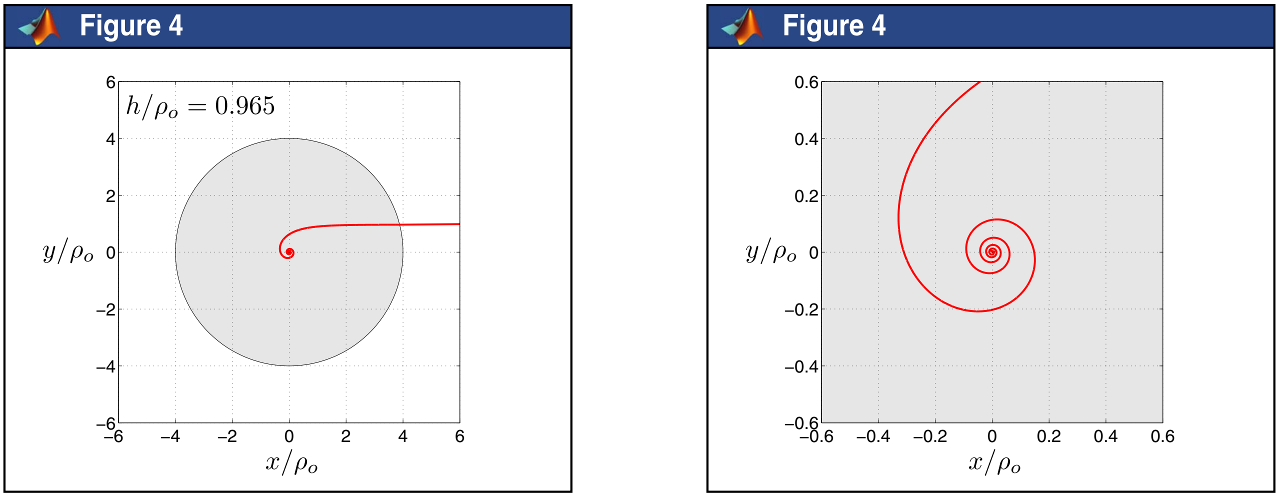

Fig. 4. Ray path for $h = 0.965\rho_o$ (central part is enlarged on the right).

Some particular situations are shown in Fig. 3 for different values of the ratio h/ρ_o.
One can think that the light can make a large number of turns before leaving the disk $\mathcal{D}$. On the contrary, below a limit for h, the path of the light identifies to that of a spiral: the ray seems to be attracted by the singularity of the refractive index of the non-homgeneous media for $\rho = 0$ and does not emerge from the disk $\mathcal{D}$. Such an amazing situation is represented in Fig. 4 for $h = 0.965\rho_o$.

IV. Conclusion

This contribution has described a MATLAB code for solving the iconal equation, especially, but not only, in media with spatially varying refractive index. The code is fast, accurate and easy to use. It has to be fed with only one function supplied by the student for switching from one study to another, the initial conditions of the problem being changed accordingly. Since the final trajectory of the light is obtained almost in real time, these initial settings can be changed at will so that an interactive study of the propagation of light is often conceivable with the computers in a classroom. In a classroom setting, the time to be allocated between the underground physics, the modeling issues and the freewheel experimentation is of course left to the teacher and may vary from one student to another. A numerical example conducted in a non-homogeneous media has demonstrated the capabilities of this code for illustrating in teaching conditions some amazing effects of the propagation of light in such media that cannot be brought to the attention of students without the aid of the computer.

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